Novel Algorithms for Nonlinear Stability Analysis with Applications to Aircraft Dynamics

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Abstract

We propose extensions to a method for estimating the region of attraction (ROA) of nonlinear ODE systems based only on measurements of convergent trajectories. We extend the ROA estimation technique to ODE systems with finitely many trim points, and to ODE systems with an arbitrary set of trim points. Additionally, we propose two novel techniques to improve the rate of convergence to the true ROA. The first technique is to partition each convergent trajectory into several trajectories for faster estimation convergence. The second is guaranteeing that divergent initial conditions are outside the estimated region of attraction by adding additional constraints to the estimation algorithm. These tools are used to analyze the longitudinal stability of the NASA GTM scale aircraft, in particular the capability of a given control strategy to return the aircraft to an equilibrium point. In this analysis, we show the feasibility of this ROA estimation strategy in a problem of dimension four.

Introduction

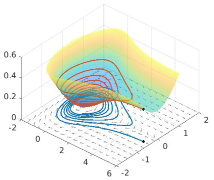
Given a dynamical system f, finding a <u>Lyapunov function V</u> proves asymptotic stability on any level set Vγ. We denote g(x,t) the solution map of f.

$$\dot{x}(t) = f(x(t))\ ,\ x(t) \in \mathbb{R}^n$$

$$V(0) = 0,\ V(x) > 0,\ \dot{V}(x) < 0$$

$$V_\gamma = \{x \in \mathbb{R}^n | V(x) \le \gamma\}$$

$$x \in V_\gamma \implies \lim_{t\to\infty} g(x,t) = 0$$

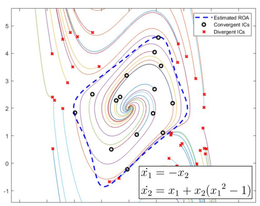


["Method of Lyapunov Functions," *Math24*.]

We are interested in finding an estimated Lyapunov function V* knowing only a partial solution map. In particular, the trajectories from a finite number of initial conditions (ICs) with discrete data, $g(x_i, j\Delta t)$. We utilized the methods of Colbert et. al [1] and expand upon the results.

Full Solution Map $g(x,t)$ ⇨ Lyapunov Function $V(x)$

Partial Solution Map $g(x_i, j\Delta t)$, $i = 1, ..., m$, $j = 1, ..., k$ ⇨ Estimated Lyapunov Fn. $V^*(x)$



ROA estimate of Van der Pol oscillator in reverse time. Replicated results from [1]

The work of Colbert considers a Lyapunov function of integrating the norm square of the full solution map. This Lyapunov function gives the important property that the <u>region of attraction (ROA)</u>, denoted S, of the system is the limit of Vγ.

$$V(x) = \int_0^\infty ||g(x,t)^2||dt \implies S = \lim_{\gamma\to\infty} V_\gamma$$

These algorithms for estimating the ROA of a nonlinear ODE system will be used to analyze the longitudinal stability of the <u>NASA GTM 5.5% scale aircraft</u>. The NASA GTM provides extensive full-envelope aerodynamic data from wind tunnel studies [3]. A piecewise polynomial model of the GTM's aerodynamic coefficients was developed by Cunis et al. [2] and will be utilized for this analysis.

Photo of the NASA GTM aircraft

$$\dot{V}_A = \frac{1}{m}\left(F\cos\alpha - \frac{1}{2}\rho S V_A^2 C_D(\alpha,\eta,\hat{q}) - mg\sin\gamma_A\right)$$

$$\dot{\gamma}_A = \frac{1}{mV_A}\left(F\sin\alpha + \frac{1}{2}\rho S V_A^2 C_L(\alpha,\eta,\hat{q}) - mg\cos\gamma_A\right)$$

$$\dot{q} = \frac{1}{I_y}\left(l_t F + \frac{1}{2}\rho S c_a V_a^2 C_m(\alpha,\eta,\hat{q}) - \frac{1}{2}\rho S V_A^2 C_Z(\alpha,\eta,\hat{q})\left(x_{cg}^{ref} - x_{cg}\right) + \frac{1}{2}\rho S V_A^2 C_X(\alpha,\eta,\hat{q})\left(z_{cg}^{ref} - z_{cg}\right)\right)$$

$$q = \dot{\alpha} + \dot{\gamma}$$

Improvements to ROA Estimation Methods

To find an estimated Lyapunov function V*, we search in the set of sum of squares (SOS) polynomials [1]. A polynomial p is SOS if and only of there exists a semidefinite matrix P such that $p(x) = Z_d(x)^T P Z_d(x)$, where $Z_d(x)$ are the monomials of x of degree d or less.

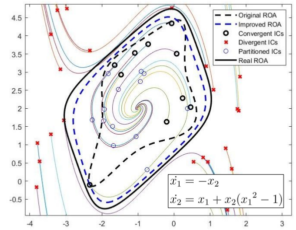
$$p(x) = \sum_{i=1}^{n} p_i(x)^2 \iff p(x) = Z_d(x)^T P Z_d(x),\ P \succeq 0$$

We search for polynomial p that maps the set of ICs x_i to the set of outputs $y_i = \log(1+V(x_i))$ and conclude $V^*(x) = 10^{p(x)} - 1$. This is done through a semi-definite positive (SDP) optimization problem, using optimization algorithm SeDuMi.

$$\{x_i \mid i = 1,\ldots,m\} \xrightarrow{p(x)} \{y_i \mid \exists x_i : y_i = \log(1 + V(x_i))\} \implies V^*(x) = 10^{p(x)} - 1$$

We propose two novel improvements to increase the speed of convergence and accuracy of the optimization:

1. **<u>Trajectory Partitioning</u>**: Consider points along known trajectories as additional initial condition, x_p, will follow the known trajectory, thus the corresponding $y_p = \log(1+V(x_p))$ is known. This allows for more initial conditions in the optimization problem without additional simulation time.
2. **<u>Avoiding Divergent ICs</u>**: The algorithm developed by Colbert et al. utilizes only convergent ICs, and the estimated ROA may contain known divergent ICs. We guarantee all known divergent ICs are outside the estimated ROA by requiring $\min\{Z_d(x_{d,i})^T P Z_d(x_{d,i})\} > \max\{Z_d(x_i)^T P Z_d(x_i)\}$ where $\{x_{d,i} \mid i = 1,\ldots,q\}$ is the set of divergent initial conditions.



$$\min_{P\in\mathbb{R}^{q\times q},\ \gamma\in\mathbb{R}^m} -\sum_{i=1}^{m}\gamma_i$$

$$\text{s.t.}\quad Z_d(x_i)^T P Z_d(x_i) - y_i \ge \gamma_i$$

$$y_i - Z_d(x_i)^T P Z_d(x_i) \ge \gamma_i\ ;\ P \succeq 0.$$

$$\min\{Z_d(x_{d_i})^T P Z_d(x_{d_i})\} > \max\{Z_d(x_i)^T P Z_d(x_i)\}$$

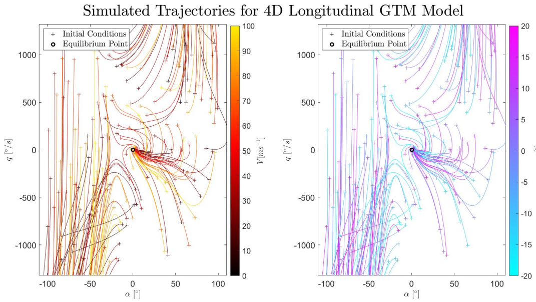
Left: Estimated ROA of Van Der Pol Oscillator without and with partitioned ICs
Top: SDP optimization problem guaranteeing divergent ICs are outside the estimated ROA.

NASA GTM Aircraft Stability Analysis

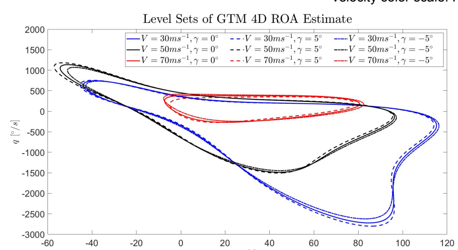
Trajectory data of the NASA GTM longitudinal dynamics were found using the piecewise polynomial model [2]. The longitudinal dynamics consist of a 4D state X, and 2D control input U. 8565 trajectories were found under constant control input η = 0rad, T = 20N. We estimate the 4D ROA and plot several level sets onto (α,q) 2D space. We further test 550 ICs with V = 50m/s, γ = 0deg, to verify the corresponding level sets.

$$\mathcal{X} = \begin{bmatrix}\alpha\\ q\\ V\\ \gamma\end{bmatrix} = \begin{bmatrix}\text{Angle of Attack (rad)}\\ \text{Pitch Rate (rad s}^{-1}\text{)}\\ \text{Velocity (m s}^{-1}\text{)}\\ \text{Inclination Angle (rad)}\end{bmatrix}$$

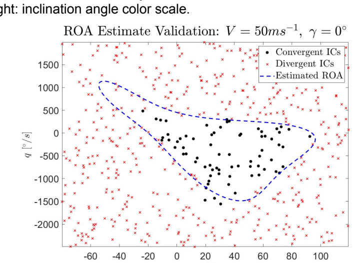
$$\mathcal{U} = \begin{bmatrix}\eta\\ F\end{bmatrix} = \begin{bmatrix}\text{Elevator Deflection (rad)}\\ \text{Thrust (N)}\end{bmatrix}$$



NASA GTM longitudinal dynamics simulations with fixed control: 0deg elevator deflection, 20N thrust. Plot of 2% of 8565 trajectories used in ROA estimate. Left: velocity color scale. Right: inclination angle color scale.



Level sets of 4D estimated NASA GTM longitudinal dynamics ROA in state space X.



550 ICs additional simulated with starting V = 50m/s, γ = 0deg and plotted against corresponding level set.

Extension to General ODE systems

The ROA estimation method presented in [1] is additionally limited to systems of a single equilibrium point. It is of interest to further consider general ODE systems. First, we consider a system with finitely many equilibrium points, and propose a method to find the ROA S_{A1} to a subset A_1 of equilibrium points.

$$f(x_1) = \ldots = f(x_q) = 0\ ;\ A_1 \subseteq \{x_1,\ldots,x_q\}$$

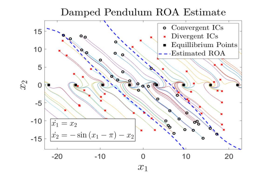
$$S_{A_1} = \{x \in \mathbb{R}^n \mid \exists x_i \in A_1\ s.t.\ \lim_{t\to\infty} g(x,t) = x_i\}$$

To find S_{A1} we consider SOS polynomials $p_{A1}(x)$, defined below, that are zero at each equilibrium point. This method is best for estimating the ROA to a small subset of equilibrium points as problem complexity increases drastically with more equilibrium points. Estimating the ROA of an ODE system to two equilibrium points was used as a proof of concept.

$$C_d(A_1) = \left\{C \subseteq \bigcup_{x_{(i)}\in A_1} X_{(i)}\ :\ |C| \le d,\ C \cap X_{(i)} \neq \varnothing\ \ \forall X_{(i)}\right\}$$

$$Z_{d,A_1}(x) = \text{The vector of all} \prod_{c\in C} c,\ \text{where } C \in C_d(A_1)\quad ;\quad p_{A_1} = Z_{d,A_1}(x)^T P Z_{d,A_1}(x)$$

Next, we consider a more general ODE, with an arbitrary set of equilibrium points, and finitely many limit cycles. We denote E the set of equilibrium points of f, C the set of limit cycles of f, and A_2 a subset of their union. We are interested in the ROA S_{A2} to this subset A_2. Although not tested on an example system, we present an algorithm to estimate S_{A2}.



Estimated ROA to equilibrium points (-π,0) and (π,0) for the damped pendulum ODE

$$E = \{x \mid f(x) = 0\},\ C = \{\text{limit cycles } c \text{ of } \dot{x} = f(x)\},$$

$$\bar{C} \subseteq C,\ \bar{E} \subseteq E,\ A_2 = \bar{E} \cup (\bigcup_{\bar{c}\in\bar{C}} \bar{c})$$

$$S_{A_2} = \{x \in \mathbb{R}^n \mid \lim_{t\to\infty} g(x,t) \in \bar{E} \text{ or } \exists \bar{c} \in \bar{C}\ s.t.\ \lim_{t\to\infty} g(x,t) \in \bar{c}\}$$

Begin by considering p_{A2} of the form p_{A1} for a finite subset of points in A_2. We assume the set A_2 is either known or able to be approximated. Construct a continuously differentiable, lipschitz function r that is 0 on A_2 and positive elsewhere.

$$\bar{A}_2 \subset A_2,\ |\bar{A}_2| < \infty\quad ;\quad p_{A_2}(x) = Z_{d,\bar{A}_2}^T(x) P Z_{d,\bar{A}_2}(x)$$

$$\{x_i \mid i = 1,\ldots,m\} \xrightarrow{p_{A_2}(x)} \left\{\log(1 + \frac{V(x_i)}{r(x_i)}) \mid i = 1,\ldots m\right\}$$

$$\implies V^*(x) = 10^{p_{A_2}(x)r(x)} - 1,\ S_{A_2} \approx V_{\gamma^*}$$

We make the above mapping via. the same SDP optimization problem. As r(x) is constructed to be positive everywhere except A_2, V* satisfies V* > 0. By the defining V as the time integral of the solution map g(x,t), $\dot{V}$ < 0 follows. Thus for an optimized p_{A2}, $\dot{V}^*$ < 0. So, V* is indeed a Lyapunov function and S_{A2} is approximated.

Conclusion

The major results of this work are improvements to the ROA estimation algorithm [1], extension of this technique to ODEs of general equilibrium sets, and application to a 4D aircraft dynamics example. We plan to further study the ability of an upset recovery strategy to return an aircraft to linear dynamics using this approach. A major limitation of this approach is SDP optimization failure in higher dimensions using conventional SDP optimization software. Given optimization difficulties in our 4D analysis, we suspect optimization difficulty in most problems dimension 5 and higher.

Abridged References

[1] Colbert, B. and M. Peet, "Using Trajectory Measurements to Estimate The Region of Attraction of Nonlinear Systems," Proceedings of the 2018 IEEE Conference on Decision and Control, pp.2341-2347, 2018.
[2] Torbjørn Cunis, Laurent Burlion, Jean-Philippe Condomines. Piecewise Polynomial Model of the Aerodynamic Coefficients of the Generic Transport Model and its Equations of Motion. [Technical Report] Third corrected version, ONERA – The French Aerospace Lab; École Nationale de l'Aviation Civile. 2018
[3] "Flight Dynamics Simulation of a Generic Transport Model," 2016.URL https://software.nasa.gov/software/LAR-17625-1.