A Divisibility Property for the Fibonacci Sequence

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Abstract

The Fibonacci sequence consists of numbers in which each number is the sum of the two preceding numbers, starting with 1 and 1. This was invented by an Italian mathematician Leonardo Pisano Bigollo in 1202. Several properties of the sequence, such as divisibility, have been studied by many mathematicians. One of the well-known properties is that for every integer m, there are infinitely many numbers of the sequence that are divisible by m, and at least one can be found among the first $2m$ numbers of the sequence. In this project, we focused on the generalization of this property. We conjectured that for every integer m, there are infinitely many numbers of the sequence that are divisible by m, and at least r can be found among the first $2rm$ numbers of the sequence. We numerically verified this conjecture using SageMath and Microsoft Excel, with the first 7,000 numbers of the sequence generated using Java.

A Known Divisibility Property

For every integer m, there are infinitely many numbers of the sequence that are divisible by m, and at least one can be found among the first $2m$ numbers of the sequence.

Numerical Illustration

This property was verified on SageMath using the first 7,000 Fibonacci numbers as illustrated in the table below.

m	1	2	3	...	10	...	101	...	2000
i	1	3	4	...	15	...	50	...	1500
F_i	1	2	3	...	610	...	12586269025	...	...

Table 1: m, i, F_i such that F_i is the 1^{st} Fibonacci number that is divisible by m.

Objectives

The main objective of this project was to answer the following questions. We obtained three conjectures by experimenting with the first 7,000 Fibonacci Numbers on SageMath.

1. Given integer m, how far in the Fibonacci sequence do we need to search to find two numbers of the sequence divisible by m?
2. Given integer m, how far in the Fibonacci sequence do we need to search to find r numbers of the sequence divisible by m?
3. Are there certain integers m such that at least one Fibonacci number divisible by m can be found among the first m numbers of the sequence?

Conjectures

1. **For every integer m, there are infinitely many numbers of the sequence that are divisible by m, and at least two can be found among the first $4m$ numbers of the sequence.**

m	1	2	3	...	10	...	101	...	850	...	1087
i	2	6	8	...	30	...	100	...	450	...	128
Fi	1	8	21	...	832040	...	...	...	...	...	...

Table 2: m, i, F_i such that F_i is the 2^{nd} Fibonacci number that is divisible by m.

2. **For every integer m, there are infinitely many numbers of the sequence that are divisible by m, and at least r can be found among the first $2rm$ numbers of the sequence.**

m	1	2	3	...	10	...	101	...	894	...	923
i	3	9	12	...	45	...	150	...	444	...	210
Fi	2	34	144	...	1134903170	...	...	...	...	...	...

Table 3: m, i, F_i such that F_i is the 3^{rd} Fibonacci number that is divisible by m.

m	1	2	3	...	10	...	101	...	700	...	754
i	4	12	16	...	60	...	200	...	2400	...	168
F_i	3	144	987	...	1548008755920	...	...	...	...	...	...

Table 4: m, i, F_i such that F_i is the 4^{th} Fibonacci number that is divisible by m.

m	1	2	3	...	10	...	101	...	600	...	693
i	5	15	20	...	75	...	250	...	300	...	210
Fi	5	610	6765	...	...	...	...	...	...	...	...

Table 5: m, i, F_i such that F_i is the 5th Fibonacci number that is divisible by m.

3. **For integer m, at least one Fibonacci number divisible by m can be found among the first m numbers of the sequence, if m is any of the following forms for any natural number.**
   - *i.* $m = 8n$
   - *ii.* $m = 12n$
   - *iii.* $m = 18n$
   - *iv.* $m = 28n$
   - *v.* $m = 44n$

Remarks

1. Tables 3-5 illustrate the second conjecture for $r = 3,4,5$.
2. Higher Fibonacci numbers in all the tables are denoted by "..." due to their sizes.

These conjectures were numerically verified on SageMath using the first 7,000 Fibonacci Numbers. The following code is for the case $m = 8n$ in the third conjecture.

```
for n in [1,..,700]:
  m=8*n
     i=1
     while(Fn[i]%m!=0):
         i=i+1
     print(n,'\t',m,'\t',i)
     i=1
```

More on Conjecture 3

There are few forms of m for which the Conjecture 3 is not true, since the first Fibonacci number divisible by m is not among the first m numbers.

Forms of m	m	i	F_i
$2n$	2	3	2
$3n$	3	4	3
$5n$	10	15	610
$4n-1$	3	4	3
$5n-1$	4	6	8
$6n-1$	23	24	46368
n^2	4	6	8

Table 6: m, i, F_i such that F_i is the 1^{st} Fibonacci number that is divisible by m.

Conclusion

The Fibonacci sequence has many well-known interesting properties. In this project, we created three conjectures. The first two conjectures are generalization of the given property. The third conjecture provides a few classes of integers that satisfy a certain divisibility property. As an extension of third conjecture, we found some classes of integers that do not satisfy the property. All three conjectures were verified numerically.

One of the future plans includes proving these conjectures. In addition, we continue to investigate more properties of the sequence.

Reference

Gardner, M. (1992). *Mathematical circus*. American Mathematical Society.

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