A Property of Fibonacci Sequence Involving Determinant

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Abstract

The Fibonacci sequence was published by Leonardo Bonacci in his book *Liber Abaci* in 1202. Each successive term in the sequence is recursively defined as the sum of the preceding two terms, starting from one and one. There are many applications of this sequence. While numerous properties and identities related to this sequence have been already identified and discovered to a great extent, the search for undiscovered properties still stands today as an interesting area of exploration in applied as well as theoretical mathematics. In this project, we studied properties of the Fibonacci sequence. We generated the first 7,000 terms of the sequence using Java. We used SageMath extensively to investigate the properties of the sequence numerically. With the help of some well-known properties, we established a new property which involves determinant of a matrix containing terms of the Fibonacci sequence. This work was supported by the New Jersey Space Grant Consortium Academic Year Internship.

Known Properties

1. **Cassini's identity**: $\det\left(\begin{bmatrix} F_{p+1} & F_p \\ F_p & F_{p-1} \end{bmatrix}\right) = (-1)^p$
2. **d'Ocagne's Identity**: $\det\left(\begin{bmatrix} F_{q+1} & F_q \\ F_{p+1} & F_p \end{bmatrix}\right) = (-1)^q F_{p-q}$
3. **Catalan's Identity**: $\det\left(\begin{bmatrix} F_p & F_{p+q} \\ F_{p-q} & F_p \end{bmatrix}\right) = (-1)^{p-q} {F_q}^2$

Definition

For $r, s \in \mathbb{N}$ with $r \geq 2$, let us define a **matrix determinant A_{rs}** as follows:

$$A_{rs} = det\left(\begin{bmatrix} F_r & F_{s(r+1)} \\ F_{s(r-1)} & F_r \end{bmatrix}\right).$$

New Results

Theorem 1:

Let $r, s \in \mathbb{N}$ with $r \geq 2$ and $s \geq 2$. If $A_{sr} = A_{rs}$, then r and s have the same parity.

Theorem 2:

Let $r, s \in \mathbb{N}$ with $r \geq 2$ and $s \geq 2$. If r and s have the same parity, then $A_{sr} = A_{rs}$.

Proof of Theorem 1

Let $A_{sr} = A_{rs}$. Suppose r and s do not have the same parity.
Let $r = 2m$ and $s = 2n + 1$ for some $m, n \in \mathbb{N}$. Then,

$$A_{sr} = det\left(\begin{bmatrix} F_{(2n+1)} & F_{(2m)[(2n+1)+1]} \\ F_{(2m)[(2n+1)-1]} & F_{(2n+1)} \end{bmatrix}\right).$$

By **Catalan's identity**,

$$A_{sr} = F_{2n+1}^2 + [(-1)^{(2m)(2n)} F_{2m}^2 - F_{(2m)(2n+1)}^2].$$

Thus,

$$A_{sr} = F_{2n+1}^2 + F_{2m}^2 - F_{(2m)(2n+1)}^2.$$

Similarly,

$$A_{rs} = F_{2m}^2 - F_{2n+1}^2 - F_{(2n+1)(2m)}^2.$$

Since $A_{sr} = A_{rs}$,

$$F_{2n+1}^2 + F_{2m}^2 - F_{(2m)(2n+1)}^2 = F_{2m}^2 - F_{2n+1}^2 - F_{(2n+1)(2m)}^2$$
$$\Longrightarrow F_{(2n+1)} = 0.$$

This is a contradiction to the fact that $F_{(2n+1)} \neq 0$ for all $n \in \mathbb{N}$.
Therefore, r and s have the same parity.

Proof of Theorem 2

Let r and s have the same parity. Consider the following two cases.

Case i: r and s are both even.

Let $r = 2m$ and $s = 2n$ for some $m, n \in \mathbb{N}$. Then,

$$A_{sr} = det\left(\begin{bmatrix} F_{(2n)} & F_{(2m)[(2n)+1]} \\ F_{(2m)[(2n)-1]} & F_{(2n)} \end{bmatrix}\right).$$

By **Catalan's identity**,

$$A_{sr} = F_{2n}^2 + [(-1)^{(2m)(2n-1)} F_{2m}^2 - F_{(2m)(2n)}^2].$$

Thus,

$$A_{sr} = F_{2n}^2 + F_{2m}^2 - F_{(2m)(2n)}^2.$$

Similarly,

$$A_{rs} = F_{2m}^2 + F_{2n}^2 - F_{(2n)(2m)}^2.$$

Therefore, $A_{sr} = A_{rs}$.

Case ii: r and s are both odd.

Let $r = 2m + 1$ and $s = 2n + 1$ for some $m, n \in \mathbb{N}$. Then,

$$A_{sr} = det\left(\begin{bmatrix} F_{(2n+1)} & F_{(2m+1)[(2n+1)+1]} \\ F_{(2m+1)[(2n+1)-1]} & F_{(2n+1)} \end{bmatrix}\right).$$

By **Catalan's identity**,

$$A_{sr} = F_{2n+1}^2 + [(-1)^{(2m+1)(2n)} F_{2m+1}^2 - F_{(2m+1)(2n+1)}^2].$$

Thus,

$$A_{sr} = F_{2n+1}^2 + F_{2m+1}^2 - F_{(2m+1)(2n+1)}^2.$$

Similarly,

$$A_{rs} = F_{2m+1}^2 + F_{2n+1}^2 - F_{(2n+1)(2m+1)}^2.$$

Therefore, $A_{sr} = A_{rs}$.

Numerical Verification

The following code was used to check that there are no r and s of the same parity when $A_{sr} \neq A_{rs}$ and there are no r and s of the opposite parity when $A_{sr} = A_{rs}$.

```
file=open('Fibo7000.txt','r')
Fn=[]
line=file.readline()
while(line !=''):
   pair=line.split('\t')
   Fn.append(Integer(pair[1]))
   line=file.readline()
for r in [2,..,83]:
   for s in [2,..,83]:
      Ars=((Fn[r])^2)-(Fn[(s*(r+1))]*(Fn[(s*(r-1))]))
      Asr=((Fn[s])^2)-(Fn[(r*(s+1))]*(Fn[(r*(s-1))]))
      if Ars!=Asr and (r-s)%2==0:
            print('r and s of same parity when Ars!=Asr: ',r,s)
      if Ars==Asr and (r-s)%2!=0:
            print('r and s of opposite parity when Ars=Asr: ',r,s)
print ("done")
```

Example:

For $r = 102, s = 506$,

$$\det\left(\begin{bmatrix} F_{102} & F_{52118} \\ F_{51106} & F_{102} \end{bmatrix}\right) = \det\left(\begin{bmatrix} F_{506} & F_{51714} \\ F_{51510} & F_{506} \end{bmatrix}\right).$$

Conclusion & Future Work

In this project, we established a new result which states that $A_{sr} = A_{rs}$ if and only if r and s have the same parity for $r \geq 2$ and $s \geq 2$. Future work for this project includes investigating more properties of similar nature for the Fibonacci sequence.

References

1. Gardner, M. (1992). *Mathematical circus*. American Mathematical Society
2. Spivey, M. Z. (2006). Fibonacci identities via the determinant sum property. *The College Mathematics Journal*, *37*(4), 286-289.

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